The cosine rule

To use the sine rule, you must have a side and the opposite angle

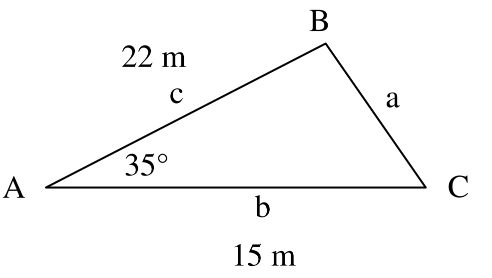
If you have:

(i) two sides and the included angle (SAS)

(ii) three sides (SSS)

The sine rule just won't work.

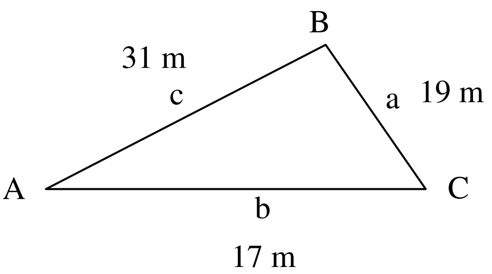
Examples:



$$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$$

SAS

We need to know 3 out of 4 values.



$$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$$

SSS

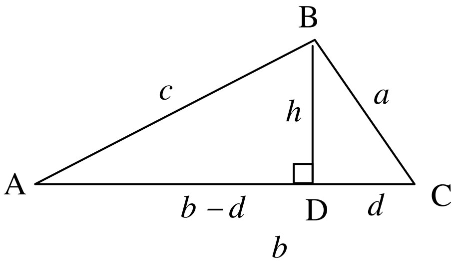
We need to know 3 out of 4 values.

We need a method for handling these two situations.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The Cosine Rule

THEORY



In triangle ABC, draw a perpendicular line from B to AC meeting AC at D.

This creates two right angled triangles ABD and BDC

Use Pythagoras in each right angled triangle

In triangle ABD:

$c^2 = h^2 + (b-d)^2$

simplifying we get

$c^2 = h^2 + b^2 - 2bd + d^2$ ... (i)

In triangle BDC:

$a^2 = h^2 + d^2$

Rearranging we get:

$d^2 = a^2 - h^2$ … (ii)

Replacing d^2 in (i) gives us

$c^2 = h^2 + b^2 - 2bd + a^2 - h^2$

which simplifies to $c^2 = b^2 - 2bd + a^2$ … (iii)

We now need to get rid of d.

Note that we can use SOH–CAH–TOA in triangle BDC

and so we can write: $\cos C = \frac{d}{a}$. Rearranging this gives us $d = a\cos C$

Substituting for d in (iii) we get:

$c^2 = b^2 - 2ba\cos C + a^2$ and now tidy this up to get:

$$c^2 = a^2 + b^2 - 2ba\cos C$$

Since we can do this round the triangle from each vertex, this can cyclically permute the letters.

So we get the usual form:

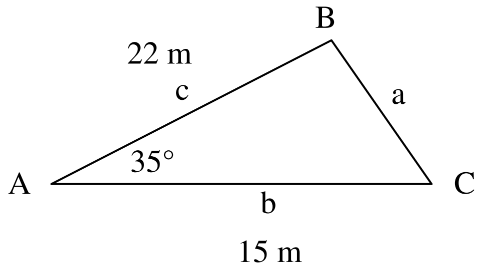
$$a^2 = b^2 + c^2 - 2bc\cos A$$

This is known as the **cosine rule**.

Cyclically permuting we get: $b^2 = c^2 + a^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$

The Cosine Rule

Examples:



SAS

Using the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

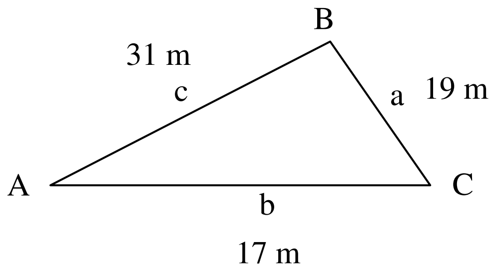
put in the values: $a^2 = 15^2 + 22^2 - 2\times 15\times 22\times \cos 35$

$a^2 = 225 + 484 - 660\times \cos 35$

$a^2 = 168.36 \quad \Rightarrow \quad a = \sqrt{168.36}$

$a = 12.98$ metres

Example



SSS

Which formula do we use for the cosine rule ?

It depends on which angle we are trying to find. Let us try to find angle B.

Then: $b^2 = c^2 + a^2 - 2ac\cos B$

NB the formula sheet also gives this in the form: $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

We will use this formula: $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

Put in the numbers: $\cos B = \dfrac{31^2 + 19^2 - 17^2}{2\times 19\times 31}$

Use the calculator: $\cos B = \dfrac{1033}{1178}$ so $B = \cos^{-1}\left(\dfrac{1033}{1178}\right) = 28.7°$

The Cosine Rule

Negative sign in cos B

You should note, (for reasons that you will learn about later), that

if **cos *B* < 0**   i.e. negative,
then ignore the negative sign and subtract the angle you get from 180°

Example: $\cos B = -0.45$

Then: (i)   ignore the negative sign.

(ii)   Find the acute angle using $B = \cos^{-1} 0.45 = 63.3°$

(iii)   Subtract this from 180.   B = 180 – 63.3   B = 116.7°

So B is an **obtuse** angle, if cos B is **negative**.

Past Paper Question

The bonnet of a car is held open, at an angle of 57°, by a metal rod.

In the diagram,

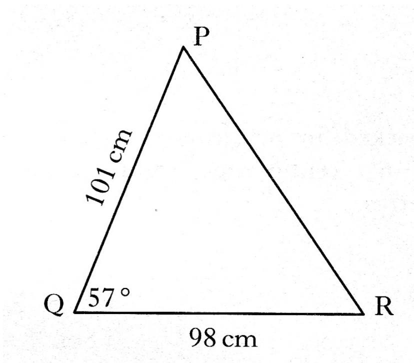
PQ represents the bonnet
PR represents the metal rod.

QR represents the distance from
the base of the bonnet to the front of the car.

PQ is 101 centimetres
QR is 98 centimetres

Calculate the length of the metal rod, PR.

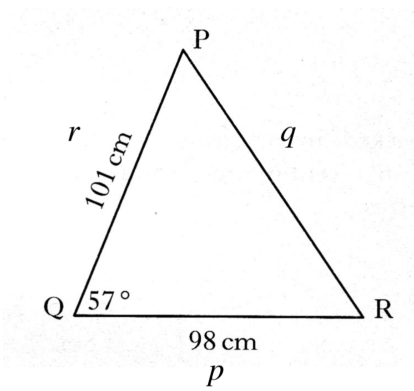
Do not use a scale drawing.



Solution:

This is **SAS** so use cosine rule.

Label the triangle sides *p, q, r*.



Then: $q^2 = r^2 + p^2 - 2rp\cos Q$

Put in numbers: $q^2 = 101^2 + 98^2 - 2 \times 101 \times 98 \times \cos 57°$

$q^2 = 19805 - 19796 \times \cos 57°$

$q^2 = 9023.32$

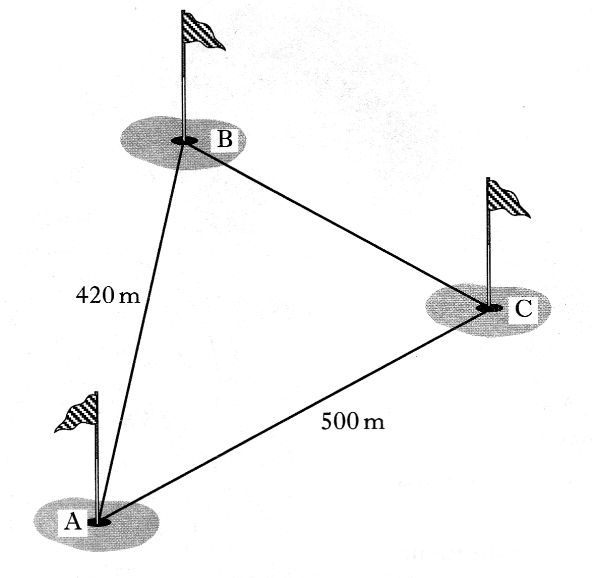
$q = \sqrt{9023.32}$ $q = 94.99$

So length of rod PR is 95 cm

The Cosine Rule

You try this one:

The diagram shows part of a golf course.



The distance AB is 420 metres,
the distance AC is 500 metres
and angle BAC = 52°.

Calculate the distance BC.

Do not use a scale drawing.

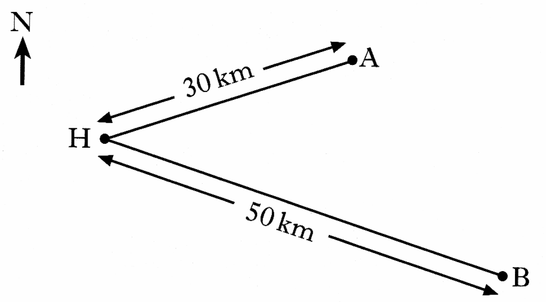
[Ans. = 409.67 m]

And another:

Two yachts leave from harbour H.

Yacht A sails on a bearing of 072°
for 30 kilometres and stops.

Yacht B sails on a bearing of 140°
for 50 kilometres and stops.



How far apart are the two yachts
when they have both stopped?

Do not use a scale drawing.

[Ans. = 47.7 km]

The Cosine Rule

Selecting a Strategy

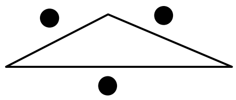
How do you know whether to use the **sine rule**, the **cosine rule** or **SOH–CAH–TOA** ?

1. Look at the triangle – is it a **right angled** triangle

   if it is – use **SOH–CAH–TOA**

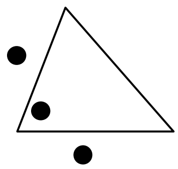
2. If it is **not right angled**:

   Are the given values



SSS

or



SAS

   if it is then **Cosine Rule.**

3. If anything else

   then **Sine Rule.**