A Brief review of Differentiation

Background to Differentiation:

Differentiation is concerned only with creating a function that allows you to calculate the gradient of a curve at any point; or to be more specific, the gradient of the tangent to the curve at any point.

This is known as the gradient function.

Differentiation of a function $y = \ldots\ldots$ gives $\frac{dy}{dx}$ the gradient function.

Differentiation of a function $f(x) = \ldots\ldots.$ gives $f'(x)$ the gradient function.

By knowing the gradient function we can apply it it to a whole host of different situations, which allows us to solve many problems.

Derivative

$\frac{dy}{dx}$ and $f'(x)$. Both of these forms are equivalent and are know as the derivative of *y* with respect to *x*; or the derivative of *f* with respect to *x* (respectively).

Gradient function

The derivative – the derived function – is called the gradient function, because it allows you to calculate the gradient of the tangent at any point *x* on the graph.

We obtain the gradient by evaluating the gradient function at the point we are interested in.

e.g. Find the gradient of the tangent to the curve $y = 2x^2 + 3x + 1$ at $x = 2$.

Differentiate to obtain the gradient function: $\frac{dy}{dx} = 4x + 3$

So the gradient at $x = 2$ is: $\frac{dy}{dx} = 4(2) + 3 = 11$

Rate of Change

The derivative – is also the rate of change of *y* with respect to *x*. It shows how quickly *y* is changing as *x* changes. A particular application of this is with velocity and acceleration.

e.g. Given $y = 3x^2 + 2x - 5$ find the rate of change of *y* with respect to *x* when $x = 4$.

Rate of change is $\frac{dy}{dx}$, so $\frac{dy}{dx} = 6x + 2$ and when $x = 4$, $\frac{dy}{dx} = 6(4) + 2 = 26$

Velocity and Acceleration

If x is the displacement of an object from an origin at time t.

Then $\frac{dx}{dt}$ is the rate of change of displacement with time – i.e. the velocity. $v = \frac{dx}{dt}$

If we differentiate the velocity and find how the velocity changes with time – we have the acceleration

So, $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$ the time t is the link between displacement (x), velocity (v) and acceleration (a).

If a projectile is fired vertically upwards, then when it reaches its maximum height, the velocity is 0.

This lets you find out the time at this instant. From this you can work out the displacement (maximum height reached) and the acceleration. If you integrate the acceleration, you have the velocity equation. If you integrate the velocity equation you get the displacement equation.

e.g. A ball is thrown vertically upwards and its height above the ground, h metres, after t seconds, is given by

$$h = 2 + 19.6t - 4.9t^2$$

What is the maximum height reached and how long does it take to get there.

When the ball reaches maximum height, it stops. i.e. the velocity = 0

Hence $v = \frac{dh}{dt} = 19.6 - 9.8t$ since at maximum height, $v = 0$, then

$$19.6 - 9.8t = 0 \quad \rightarrow \quad 9.8t = 19.6 \quad \rightarrow \quad t = 2$$

So the ball goes up for 2 seconds.

The maximum height reached is after 2 seconds.

So to find this height, go back to the original equation:

$$h = 2 + 19.6t - 4.9t^2 \quad \rightarrow \quad h = 2 + 19.6(2) - 4.9(2)^2 \quad \rightarrow \quad 2 + 39.4 - 19.6 \quad = \quad 21.8 \ metres$$

Optimisation – Problem Solving.

Often we want to find the dimensions that will maximise or minimise the surface area or the volume of an object such as a container, or the maximum area enclosed by a boundary fence, or a minimum perimeter.

Differentiation can help here. If we have a function that describes the surface area, the volume or the perimeter, then we want to find the maximum or minimum value of the function and the value of the variable (the measurement) where it occurs.

We form an expression for the Area, or Volume, or perimeter.

We reduce the problem to only one variable, by using a given constraint in the problem.

Differentiate to find the Stationary points.

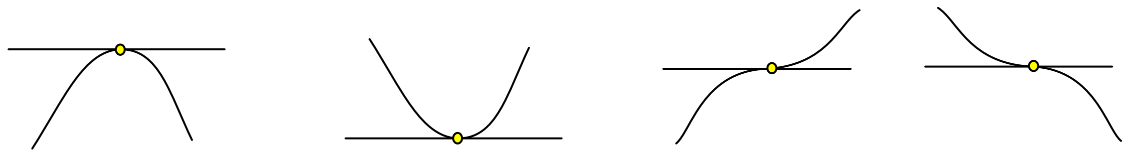
Determine whether maximum or minimum using the table of signs.

Calculate the other dimensions if asked for in the question.

Answer the question with a sentence. [**See separate tutorial on Optimisation and Problem Solving**].

Stationary Points

A stationary point is where the curve is neither increasing nor decreasing. In other words it is 'stationary'. The gradient is 0.



maximum　　　minimum　　　points of inflexion

function is always increasing　　　function is always decreasing

So for a SP (stationary point) $\frac{dy}{dx} = 0$ (or $f'(x) = 0$ using function notation)

This allows us to find the *x*-coordinate of stationary points on a curve. Since we usually want the coordinates, we substitute this value of *x* into the original equation $y = ...$ or $f(x) = ...$ to find the *y*-coordinate of this point.

We now have to identify the nature of the stationary point

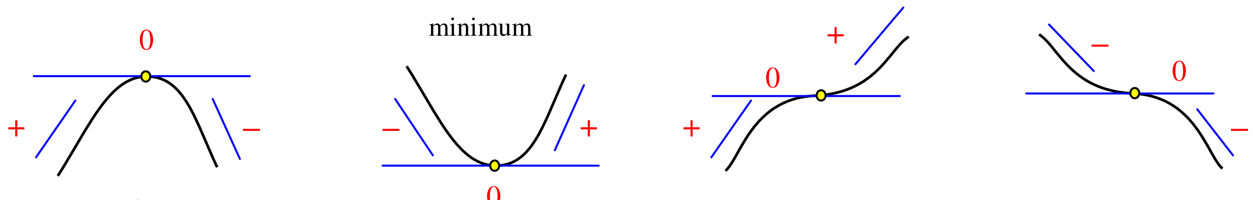
– is it a **maximum**, a **minimum** or a **point of inflexion** ?

Table of signs

We use the table of signs.

The table of signs allows us to look at the gradient on either side of the point, to see whether it is positive or negative, i.e. the function is increasing or decreasing.

We look at the sign of: $\frac{dy}{dx} \left(or\ f'(x)\right)$



maximum　　　points of inflexion

	→	SP	→
$\frac{dy}{dx}$	+	0	–
grad	/	—	\

	→	SP	→
$\frac{dy}{dx}$	–	0	+
grad	\	—	/

	→	SP	→
$\frac{dy}{dx}$	+	0	+
grad	/	—	/

	→	SP	→
$\frac{dy}{dx}$	–	0	–
grad	\	—	\

Rules for Differentiation

$y = k$ (a constant)	$\frac{dy}{dx} = 0$
$y = kx$	$\frac{dy}{dx} = k$
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$

$y = kx^n$	$\frac{dy}{dx} = nkx^{n-1}$
$y = f(x) + g(x)$	$\frac{dy}{dx} = f'(x) + g'(x)$

Chain Rule

$y = (ax+b)^n$	$\frac{dy}{dx} = n(ax+b)^{n-1} \times a$

Trigonometric Functions

$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$

Chain Rule for Trigonometric Functions

$y = \sin ax$	$\frac{dy}{dx} = a\cos ax$
$y = \cos ax$	$\frac{dy}{dx} = -a\sin ax$

The above are the only rules that you will need for differentiation.

All you have to do is make sure you apply the rules correctly.

Always make sure you are in **Straight line form** before differentiating.

Change back to **positive index** or **root form** befor **evaluating**.

Differentiation $\Rightarrow$ **DECREASE** the power.